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Central limit theorem for triangular arrays of Non-Homogeneous Markov chains

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Abstract

In this paper we obtain the central limit theorem for triangular arrays of non-homogeneous Markov chains under a condition imposed to the maximal coefficient of correlation. The proofs are based on martingale techniques and a sharp lower bound estimate for the variance of partial sums. The results complement an important central limit theorem of Dobrushin based on the contraction coefficient.

1 Introduction and notations

More than fifty years ago Dobrushin [3] proved a definitive central limit theorem for non-homogeneous Markov chains. His work is based on the coefficient of ergodicity which is defined by using the contraction coefficient, specifically for uniformly bounded functions. In a recent paper, Sethuraman and Varadhan [16] give a new and elegant proof of Dobrushin's result and provide a survey of the literature that was generated by it. In this paper we address a similar problem for Markov chains by using the maximal coefficient of correlation, instead of the contraction coefficient. This coefficient is more general and the results are applicable to a larger class of Markov processes. The problem is challenging, since the maximal coefficient of correlation is defined for functions that are square integrable only and many new tools have to be developed.

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Let $(\Omega, \mathcal{K}, \mathbf{P})$ be a probability space and let $\mathcal{A}, \mathcal{B}$ be two sub σ-algebras of $\mathcal{K}$. Define the maximal coefficient of correlation

$$\rho(\mathcal{A}, \mathcal{B}) = \sup_{f \in \mathbf{L}_2(\mathcal{A}), g \in \mathbf{L}_2(\mathcal{B})} |corr\,(f, g)| \ ,$$

where $\mathbf{L}_2(\mathcal{A})$ is the space of random variables that are $\mathcal{A}$ measurable and square integrable. For a vector of random variables, $(Y_k)_{1 \le k \le n}$ we define

$$\rho_k = \max_{1 \le s, s+k \le n} \rho(\sigma(Y_i, i \le s), \sigma(Y_j, j \ge s+k)) \ . \tag{1}$$

For a nonhomogeneous Markov chain of length n, $(\xi_i)_{1 \le i \le n}$, it turns out that the computation of ρ_k simplifies (see for instance Theorem 7.2 (c) in [1]). For this case,

$$\rho_k = \max_{1 \le s, s+k \le n} \rho(\sigma(\xi_s), \sigma(\xi_{s+k})) \ .$$

Moreover (see Theorem 7.4 (a) in [1]), for all $1 \le k \le n-1$,

$$\rho_k \le \rho_1^k \ .$$

In terms of the conditional expectation (see chapter 7 in [14] or Theorem 4.4 (b3) in [1]) an alternative definition of ρ_1 is

$$\rho_1 = \max_{2 \le i \le n} \sup_g \{ \frac{||\mathbf{E}(g(\xi_i)|\xi_{i-1})||_2}{||g(\xi_i)||_2} \ ; \ ||g(\xi_i)||_2 < \infty \text{ and } \mathbf{E}g(\xi_i) = 0 \} \ , \tag{2}$$

where we used the notation $||X||_p = (\mathbf{E}|X|^p)^{1/p}$, for $p > 1$.

For a stationary Markov chain defined on $(\Omega, \mathcal{K}, P)$ with values in $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ with invariant measure π and transition probability $Q(x, A) = P(\xi_1 \in A|\xi_0 = x)$, define the operator Q acting on $\mathbf{L}_2(\mathcal{X}, \mathcal{B}(\mathcal{X}), \pi)$ via

$$(Qu)(x) = \int_{\mathcal{X}} u(y) Q(x, dy) \ . \tag{3}$$

Denote $\mathbf{L}_2^0(\pi) = \{g \in \mathbf{L}_2(\mathcal{X}, \mathcal{B}(\mathcal{X}), \pi) \text{ with } \int g d\pi = 0\}$. With these notations, the coefficient ρ_1 is simply the norm operator of $Q : \mathbf{L}_2^0(\pi) \to \mathbf{L}_2^0(\pi)$,

$$\rho_1 = ||Q||_{\mathbf{L}_2^0(\pi)} = \sup_{g \in \mathbf{L}_2^0(\pi)} \frac{||Q(g)||_2}{||g||_2}. \tag{4}$$

Conditions imposed to the maximal coefficient of correlation make possible to study the asymptotic behavior of many dependent structures including classes of Markov chains and Gaussian sequences. This coefficient was used by Kolmogorov and Rozanov [9] and further studied by Rosenblatt [14], Ibragimov [6], Shao [15] among many others. An introduction to this topic, mostly in the stationary setting, can be found in the Chapters 7, 9 and 11 in Bradley [1]. Application to the central limit theorem (CLT) for various stationary Markov chains with $\rho_1 < 1$ are surveyed in Jones [8]. In the nonstationary setting and

general triangular arrays a central limit theorem was obtained by Utev [17], assuming a lower bound on the variance of partial sums and $\rho-$mixing coefficients converging to 0 uniformly at a logarithmic rate.

In this paper we are concerned with the central limit theorem for a triangular array of Markov chains. Let $(\xi_{n,i})_{1\leq i\leq n}$ be an array of non-homogeneous Markov chains defined on a probability space $(\Omega, \mathcal{K}, P)$ with values in $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. In addition, let $(f_{n,i})_{1\leq i\leq n}$ be real valued functions on $\mathcal{X}$. Define,

$$X_{n,i} = f_{n,i}(\xi_{n,i}) \text{ and } S_n = \sum_{i=1}^{n} X_{n,i} \ . \tag{5}$$

Everywhere in the paper we shall assume

$$\mathbf{E}X_{n,i} = 0,\ \mathbf{E}X_{n,i}^2 < \infty$$

and denote by

$$\sigma_n^2 = var\ S_n \text{ and } \ b_n^2 = \sum_{i=1}^{n} var\ X_{n,i}\ , \tag{6}$$

where $var\ X = \mathbf{E}(X - \mathbf{E}X)^2$. In this context, the mixing coefficients need an additional index to indicate the row. We shall write now $\rho_{n,1}$ instead of ρ_1 to specify that this coefficient is computed for $(\xi_{n,i})_{1\leq i\leq n}$.

We shall establish in Proposition 13 that for all $n \geq 1$, the quantities σ_n^2 and b_n^2 are related by the following inequality

$$\frac{1-\rho_{n,1}}{1+\rho_{n,1}} b_n^2 \leq \sigma_n^2 \leq \frac{1+\rho_{n,1}}{1-\rho_{n,1}} b_n^2\ ,$$

provided $\rho_{n,1} < 1$.

We shall further discuss the rate at which the maximal coefficients of correlation, $\rho_{n,1}$, are allowed to converge to 1, for the validity of the central limit theorem for S_n/σ_n.

The results are formulated both in terms of bounded random variables and also in an integral form similar to the Lindeberg condition. When applied to a triangular array of uniformly bounded random variables, with the variance of individual summands uniformly bounded below, our sufficient condition for the central limit theorem is implied, for instance, by

$$(1-\rho_{n,1})^3 n(\ln n)^{-2} \to \infty \text{ as } n \to \infty\ . \tag{7}$$

We can see from this result that we obtain the central limit theorem not only for the situation when $\rho_{n,1} \leq r < 1$. We can let $\rho_{n,1}$ approaches 1, but not too fast, at a rate that will be specified.

The proof of this result and the other results of this type are based on the following tools we develop in this paper that have interest in themselves:

(1) General sufficient conditions for the CLT for triangular arrays, based on a familiar projective martingale representation.

(2) Sharp lower and upper bounds for the variance of sums of variables connected in a Markov chain.

(3) Moment and exponential inequalities for certain partial sums.

Our theorems are related to Dobrushin's result which uses as a measure of dependence the contraction coefficient

$$\delta(Q) = \sup_{u \in \mathcal{U}} \sup_{x_1, x_2} |(Qu)(x_1) - (Qu)(x_2)| \ ,$$

where $\mathcal{U} = \{u, \sup_{x_1,x_2} |u(x_1) - u(x_2)| \leq 1\}$ and the operator Q is defined by (3), on the space of bounded measurable functions. For a triangular array of Markov chains $(\xi_{n,i})_{1 \leq i \leq n}$ with transition probabilities $Q_{n,i}(x, A) = P(\xi_{n,i+1} \in A | \xi_{n,i} = x)$,

$$\delta_{n,1} = \sup_{1 \leq i \leq n-1} \delta(Q_{n,i}) \text{ and } \delta_{n,k} \leq \delta_{n,1}^k. \tag{8}$$

By Lemma 4.1 in Sethuraman and Varadhan [16] we have

$$\rho_{n,1} \leq \delta_{n,1}^{1/2} \ .$$

Dobrushin [3] showed that for a triangular array of uniformly bounded random variables, with the variance of individual summands uniformly bounded below, a sufficient condition for the CLT is

$$(1 - \delta_{n,1})^3 n \to \infty \text{ as } n \to \infty \ . \tag{9}$$

Moreover, he analyzed an example attributed to Bernstein, of family of Markov chains satisfying $1 - \delta_{n,1} = n^{1/3}$ and such that the CLT fails. The initial proof of this result is very long. A simplified proof of a further reaching result can be found in Sethuraman and Varadhan [16] (see also Theorem 8). For the properties of this contraction coefficient we refer to Iosifescu and Theodorescu [7], sections 1.1. and 1.2.

There are plenty of examples for which $\delta_{n,1} = 1$, but $\rho_{n,1} < 1$, so our results have a larger sphere of applicability than the results based on $\delta_{n,1}$.

For instance, for a row-wise stationary array of Markov chains with joint distribution of $(\xi_{n,1}, \xi_{n,2})$ bivariate normal, the maximal coefficient of correlation is very simple, namely $\rho_{n,1} = |corr(\xi_{n,1}, \xi_{n,2})|$, while (if i.i.d. rows are excluded) $\delta_{n,1} = 1$ for all n (for a convenient reference to this fact see [1], Theorems 9.1 and 9.7). For functions of variables in this array our theorems are applicable, and the conditions are very easy to verify.

Moreover, even for the situation when $\delta_{n,1} < 1$, it is possible that $\rho_{n,1} \to 1$ and $\delta_{n,1} \to 1$ at different rates, such that, for instance, (7) holds but (9) does not. Such an example can be easily constructed by using a recent result by Bradley [2], who showed that for any $0 < a < b < 1$ there is a stationary Markov chain for which $\rho_1 = a$ and $\delta_1 = b$.

Our results will be useful for treating families of various Markov processes that are considered in applications. For example, Liu et all [10] have shown that if the operator induced by a Gibbs sampler satisfies a Hilbert-Schmidt condition then $\rho_1 < 1$.

Another class of examples is provided by an array of stationary reversible Markov chains that are geometrically ergodic. A stationary Markov chain is called geometrically ergodic if there is $0 < t < 1$ and a nonnegative function $M(x)$ such that $||Q^n(x, \cdot) - \pi(\cdot)|| \leq M(x)t^n$. A stationary Markov chain that is geometrically ergodic and reversible satisfies $\rho_1 < 1$. (Roberts and Rosenthal, [12]). A particular example of this kind is the popular Random Walk MHG Algorithms which are reversible by construction. In Mengersen and Tweedie [11] it was shown that the random walk samplers cannot be uniformly ergodic (so $\delta_1 = 1$) but they do establish that a random walk MHG algorithm can be geometrically ergodic in some situations, therefore they have $\rho_1 < 1$. This work was extended in Roberts and Tweedie [13].

Our paper is organized as follows: In Section 2 we state the main results. In order to prove them, in Section 3 we develop sufficient conditions for the CLT for triangular arrays of random variables based on martingale representations. Section 4 is concerned with bounds for the variance of partial sums of a Markov chain as a function of the ρ_1 coefficient. The proofs of the main results are the subject of Section 5. Some technical lemmas involving higher moments for sums and exponential bounds are postponed to the Appendix.

The convergence in probability will be denoted $\to^P$ and $\to^D$ denotes convergence in distribution.

2 Results

Our first theorem applies to triangular arrays of functions of Markov chains consisting of bounded centered variables. To describe our results it is convenient to introduce the related coefficient

$$\lambda_n = 1 - \rho_{n,1} = \min_{1 \leq s \leq n-1} [1 - \rho(\sigma(\xi_{n,s}), \sigma(\xi_{n,s+1}))] \; . \tag{10}$$

Clearly $0 \leq \lambda_n \leq 1$, and λ_n is a coefficient of independence for $(\xi_{n,i})_{1 \leq i \leq n}$ with n fixed. Notice that $\lambda_n = 1$ if and only if the vector $(X_{n,i})_{1 \leq i \leq n}$ is independent. Everywhere in this section we shall consider the nondegenerate case i.e. $\lambda_n b_n > 0$ for all $n \geq 1$. By Proposition 13 this condition is equivalent to $\lambda_n \sigma_n > 0$ for all $n \geq 1$.

Theorem 1 *Suppose that $(X_{n,i})_{1 \leq i \leq n}$ is defined by (5) and for some finite positive constants C_n we have*

$$\max_{1 \leq i \leq n} |X_{n,i}| \leq C_n \;\; a.s. \tag{11}$$

and

$$\frac{C_n(1 + |\ln(\lambda_n)|)}{\lambda_n \sigma_n} \to 0 \quad as \; n \to \infty \; . \tag{12}$$

Then

$$\frac{\sum_{i=1}^n X_{n,i}}{\sigma_n} \overset{\mathcal{D}}{\to} N(0,1) \;\; as \;\; n \to \infty \; . \tag{13}$$

We state a corollary which combines Theorem 1 with the bound of the variance given in Proposition 13 in Section 4.

Corollary 2 *Suppose that* $(X_{n,i})_{1\leq i\leq n}$ *is defined by (5). Assume that (11) holds and*

$$\frac{C_n(1+|\ln(\lambda_n)|)}{\lambda_n^{3/2}b_n}\to 0 \text{ as } n\to\infty \ .$$

Then the CLT (13) holds.

Next, we give a corollary that can be applied to an array of uniformly bounded random variables.

Corollary 3 *Suppose that* $(X_{n,i})_{1\leq i\leq n}$ *is defined by (5) and assume that there are two positive constants* C *and* c *such that* $\max_{1\leq i\leq n}|X_{n,i}|\leq C$ *a.s. and also var* $X_{n,i}\geq c>0$ *for all* $n\geq 1$ *and* $1\leq i\leq n$*. Then CLT (13) holds provided*

$$\lambda_n^3 n(1+|\ln(\lambda_n)|)^{-2}\to\infty. \tag{14}$$

Notice that (7) implies (14).

We shall also prove an integral form of Theorem 1.

Corollary 4 *Suppose that* $(X_{n,i})_{1\leq i\leq n}$ *is defined by (5) and for every* $\varepsilon>0$

$$\frac{1}{\lambda_n\sigma_n^2}\sum_{i=1}^{n}\mathbf{E}X_{n,i}^2 I(|X_{n,i}|>\varepsilon h(\lambda_n)\sigma_n)\to 0 \text{ as } n\to\infty \tag{15}$$

where $h(\lambda_n)=\lambda_n(1+|\ln(\lambda_n)|)^{-1}$*. Then the CLT (13) holds.*

We now point out some immediate consequences of Corollary 4.

Remark 5 *Let us notice that by using the bounds on the variance given in Proposition 13, condition (15) is implied by*

$$\frac{1}{\lambda_n^2 b_n^2}\sum_{i=1}^{n}\mathbf{E}X_{n,i}^2 I(|X_{n,i}|>\varepsilon h'(\lambda_n)b_n)\to 0 \text{ as } n\to\infty \tag{16}$$

where $h'(\lambda_n)=\lambda_n^{3/2}(1+|\ln(\lambda_n)|)^{-1}$.

The next remark applies to triangular arrays of Markov chains with uniformly bounded $\rho_{n,1}-$mixing coefficients.

Remark 6 *Suppose that* $(X_{n,i})_{1\leq i\leq n}$ *is defined by (5) and there is a positive number* ρ *such that* $\sup_n\rho_{n,1}\leq\rho<1$*. Then, the CLT (13) holds provided that for every* $\varepsilon>0$

$$\frac{1}{b_n^2}\sum_{i=1}^{n}\mathbf{E}X_{n,i}^2 I(|X_{n,i}|>\varepsilon b_n)\to 0 \ .$$

For arrays of Markov chains that are row-wise strictly stationary with the same invariant distribution, Corollary 4 (via Remark 5) has a simple form. Examples of this type are arrays of Markov chains generated by parametric copulas.

Remark 7 *Suppose for each n, $(\xi_{n,i})_{1\leq i\leq n}$ is a stationary Markov chain with the same invariant distribution π and transition operator Q_n. Let $f \in \mathbf{L}_2^0(\pi)$ and define $X_{n,k} = f(\xi_{n,k})$ and $\lambda_n = 1 - ||Q_n||_{\mathbf{L}_2^0(\pi)}$. Then, the CLT (13) holds provided that for every $\varepsilon > 0$*

$$\frac{1}{\lambda_n^2}\int f^2(x)I(|f(x)| > \varepsilon\sqrt{n}h'(\lambda_n))d\pi \to 0 \text{ as } n \to \infty \ .$$

Finally, we mention the extension of Dobrushin's CLT obtained by Sethuraman and Varadhan [16], by using the coefficient $\delta_{n,1}$:

Theorem 8 *Suppose that $(X_{n,i})_{1\leq i\leq n}$ is defined by (5) and relation (11) holds. Denote $\alpha_n = 1 - \delta_{n,1}$. If*

$$\frac{C_n^2}{\alpha_n^3 b_n^2} \to 0 \text{ as } n \to \infty \ , \tag{17}$$

then, the CLT (13) holds. When $|X_{n,i}| \leq C < \infty$ a.s and var $X_{n,i} \geq c > 0$, for all $1 \leq i \leq n$ and $n \geq 1$, then the CLT (13) holds provided

$$\alpha_n^3 n \to \infty \text{ as } n \to \infty \ .$$

3 Central Limit Theorem for triangular arrays

The following theorem is a variant of Theorem 3.2 in Hall and Heyde [5] (see also Gänssler and Häusler [4]).

Theorem 9 *Assume $(D_{n,i})_{1\leq i\leq n}$ is an array of square integrable martingale differences adapted to an array $(\mathcal{F}_{n,i})_{1\leq i\leq n}$ of nested sigma fields. Suppose*

$$\mathbf{E}(\max_{1\leq j\leq n}|D_{n,j}|) \to 0 \text{ as } n \to \infty \tag{18}$$

and

$$\sum_{j=1}^{n} D_{n,j}^2 \to^P 1 \quad \text{as } n \to \infty \ .$$

Then $S_n = \sum_{j=1}^n D_{n,j}$ converges in distribution to a standard normal variable.

In this section we shall assume the following general setting:

$(C1)$ Assume $(X_{n,j})_{1\leq j\leq n}$ is an array of centered random variables that are square integrable and adapted to an array of sigma fields $(\mathcal{F}_{n,j})_{1\leq j\leq n}$, with $\mathcal{F}_{n,j} \subset \mathcal{F}_{n,j+1}$ for all $1 \leq j \leq n-1$. Extend the array with $X_{n,0} = 0$ and $\mathcal{F}_{n,0} = \{\varnothing, \Omega\}$ for all n.

With this notation, as an immediate consequence of Theorem 9, we formulate:

Corollary 10 *Assume* $(C1)$ *and let* $\mathbf{E}S_n^2 = 1$. *Define the projector operator*

$$\mathbf{P}_{n,j}Y = \mathbf{E}(Y|\mathcal{F}_{n,j}) - \mathbf{E}(Y|\mathcal{F}_{n,j-1}) \ .$$

Assume

$$\max_{1\le j\le n} |\mathbf{P}_{n,j}S_n| \to^P 0 \text{ as } n\to\infty \ . \tag{19}$$

and

$$\sum_{j=1}^{n} (\mathbf{P}_{n,j}S_n)^2 \to^P 1 \text{ as } n\to\infty \ . \tag{20}$$

Then the CLT in (13) holds.

Proof. Because we assume $\mathcal{F}_{n,0} = \{\varnothing, \Omega\}$, we can express S_n in terms of projections

$$S_n = \sum_{j=1}^{n} X_{n,j} = \sum_{j=1}^{n} \mathbf{E}(S_n|\mathcal{F}_{n,j}) - \mathbf{E}(S_n|\mathcal{F}_{n,j-1}) = \sum_{j=1}^{n} \mathbf{P}_{n,j}S_n \ . \tag{21}$$

Notice that we have written S_n as a sum of martingale differences

$$d_{n,j} = \mathbf{E}(S_n|\mathcal{F}_{n,j}) - \mathbf{E}(S_n|\mathcal{F}_{n,j-1}) = \mathbf{P}_{n,j}S_n = \mathbf{P}_{n,j}(S_n - S_{j-1}) \tag{22}$$

and we apply Theorem 9. Since $\sum_{j=1}^{n} \mathbf{E}(\mathbf{P}_{n,j}S_n)^2 = \mathbf{E}S_n^2 = 1$, it follows that $\max_{1\le j\le n} |\mathbf{P}_{n,j}S_n|$ is uniformly integrable in $\mathbf{L}_1$ and then (19) implies (18). ◊

Analyzing the conditions of Corollary 10 is leading us to the following useful theorem. For $0 \le j \le n$ denote

$$A_{n,j} = \mathbf{E}(S_n - S_{n,j}|\mathcal{F}_{n,j}) \ , \tag{23}$$

where $S_{n,j} = \sum_{i=1}^{j} X_{n,i}$

Theorem 11 *Assume* $(C1)$ *and* $\mathbf{E}S_n^2 = 1$. *Also assume that*

$$\max_{1\le j\le n} (|X_{n,j}| + |A_{n,j}|) \to^P 0 \text{ as } n\to\infty \tag{24}$$

and

$$\sum_{j=1}^{n} (X_{n,j}^2 + 2X_{n,j}A_{n,j}) \to^P 1 \text{ as } n\to\infty \ . \tag{25}$$

Then S_n *converges in distribution to* $N(0,1)$.

Proof. For simplicity we drop the index n in the notation, so $X_j = X_{n,j}$, $A_j = A_{n,j}$, $d_j = d_{n,j}$.

Condition (19) follows from condition (24) since, by definitions (22) and (23),

$$|d_j| \le |X_j| + |A_j| + |A_{j-1}| \text{ a.s.}$$

To verify condition (20), for $1 \le j \le n$ let us compute

$$\begin{aligned} d_j^2 &= (X_j + A_j - A_{j-1})^2 \\ &= (X_j^2 + 2X_jA_j) + A_j^2 - A_{j-1}^2 + 2(A_{j-1} - X_j - A_j)A_{j-1} \ , \end{aligned}$$

whence, by definition (22) and the fact that $A_0 = A_n = 0$, we obtain

$$\sum_{j=1}^n d_j^2 = \sum_{j=1}^n (X_j^2 + 2X_jA_j) - 2\sum_{j=1}^n d_jA_{j-1} \ .$$

Now, by condition (25), the first term in the right hand side is converging in probability to 1. For the martingale transform $\sum_{j=1}^n d_jA_{j-1}$ we use a truncation argument. Let $\varepsilon > 0$ and denote $A_j^\varepsilon = A_jI(|A_j| \le \varepsilon)$. For any $a > 0$,

$$\mathbf{P}(|\sum_{j=1}^n d_jA_{j-1}| > a) \le \mathbf{P}(\max_{1\le j\le n} |A_j| > \varepsilon) + \mathbf{P}(|\sum_{j=1}^n d_jA_{j-1}^\varepsilon| > a)$$

$$\le \mathbf{P}(\max_{1\le j\le n} |A_j| > \varepsilon) + \varepsilon^2\mathbf{E}(\sum_{j=1}^n d_j)^2/a^2 = \mathbf{P}(\max_{1\le j\le n} |A_j| > \varepsilon) + \varepsilon^2/a^2.$$

where on the last line we used the fact that $\mathbf{E}(\sum_{j=1}^n d_j)^2 = 1$. Then, we take into account that $\max_{1\le j\le n} |A_j|$ is negligible in probability by (24) and we conclude the convergence to 0 by letting $n \to \infty$ followed by $\varepsilon \to 0$. It follows that

$$\sum_{j=1}^n d_j^2 \to^P 1 \text{ as } n \to \infty$$

and the CLT holds by Corollary 10. $\Diamond$

Theorem 11 has the following simple Corollary that will be used in our proofs:

Proposition 12 *Assume* $(C1)$ *and the variables have finite moments of order* 4*. Moreover assume the following conditions hold:*

$$\frac{1}{\sigma_n^4}\sum_{j=1}^n \mathbf{E}X_{n,j}^4 \to 0 \ \textit{as}\ n \to \infty \ . \tag{26}$$

For every $\varepsilon > 0$

$$\mathbf{P}(\max_{1\le j\le n} |A_j| > \varepsilon\sigma_n) \to 0 \ \textit{as}\ n \to \infty \tag{27}$$

and

$$\frac{1}{\sigma_n^4} var \sum_{j=1}^n (X_{n,j}^2 + 2X_{n,j}A_{n,j}) \to 0 \ \textit{as}\ n \to \infty \ . \tag{28}$$

Then $\sigma_n^{-1}S_n$ *converges in distribution to* $N(0,1)$ *.*

Proof. Conditions (26) and (27) easily imply (24). Then, condition (28) implies (25) by taking into account that

$$\frac{1}{\sigma_n^2}\mathbf{E}\sum_{j=1}^{n}(X_{n,j}^2 + 2X_{n,j}A_{n,j}) = 1 \ .$$

$\Diamond$

4 Bounds for the variance of partial sums of Markov chains

In this section we establish sharp upper and lower bounds for the variance of partial sums of a Markov chain as a function of the maximal coefficient of correlation defined in (2).

Proposition 13 *Let $(X_1, X_2, ..., X_n)$ be a vector of square integrable centered random variables that are functions of a Markov process $(\xi_i)_{1\leq i\leq n}$ i.e. $X_k = f_k(\xi_k)$. Denote by $S_n = \sum_{i=1}^n X_i$ and $\mathcal{F}_j = \sigma(\xi_i,\ i \leq j)$. We set $X_0 = 0$ and $\mathcal{F}_0 = \{0, \Omega\}$. If $\rho_1 < 1$, then*

$$\frac{1-\rho_1}{1+\rho_1}\sum_{i=1}^{n}\mathbf{E}X_i^2 \leq \mathbf{E}S_n^2 \leq \frac{1+\rho_1}{1-\rho_1}\sum_{i=1}^{n}\mathbf{E}X_i^2 \ .$$

Proof. We prove first the lower bound. For this proof we recall the notation (6) and (23), which in this case is $A_j = A_{n,j} = \mathbf{E}(S_n - S_j|\xi_j)$. We start as before from the martingale decomposition

$$S_n = \sum_{j=1}^{n}\mathbf{E}(S_n|\mathcal{F}_j) - \mathbf{E}(S_n|\mathcal{F}_{j-1}) = \sum_{j=1}^{n}\mathbf{P}_j(S_n - S_{j-1}) \ .$$

By the orthogonality of the martingale differences

$$\sigma_n^2 = \sum_{j=1}^{n}\mathbf{E}[\mathbf{P}_j(S_n - S_{j-1})]^2 \ . \tag{29}$$

Notice that by taking into account the Markov property and simple algebra

$$\begin{aligned}\mathbf{E}[\mathbf{P}_j(S_n - S_{j-1})]^2 &= \mathbf{E}(X_j + A_j)^2 + \mathbf{E}(A_{j-1})^2 \\ &\quad -2\mathbf{E}(X_j + A_j)A_{j-1} \ .\end{aligned}$$

By the definition of ρ_1

$$\begin{aligned}2|\mathbf{E}(X_j + A_j)A_{j-1}| &\leq 2\rho_1||X_j + A_j||_2||A_{j-1}||_2 \\ &\leq \rho_1^2\mathbf{E}(X_j + A_j)^2 + \mathbf{E}(A_{j-1})^2 \ ,\end{aligned}$$

which combined with the previous identity gives

$$\mathbf{E}[\mathbf{P}_j(S_n - S_{j-1})]^2 \geq (1-\rho_1^2)\mathbf{E}(X_j + A_j)^2.$$

Therefore, by summing these inequalities we obtain

$$\sigma_n^2 \geq (1-\rho_1^2)\sum_{i=1}^{n}\mathbf{E}(X_i + A_i)^2. \tag{30}$$

On the other hand, by (29) and the properties of conditional expectation

$$\sigma_n^2 = \sum_{i=1}^{n}\mathbf{E}(X_i + A_i)^2 - \sum_{i=1}^{n}\mathbf{E}A_{i-1}^2 \ .$$

By introducing this identity in relation (30) and changing the variable of summation we obtain

$$\sigma_n^2 \geq (1-\rho_1^2)[\sum_{i=1}^{n}\mathbf{E}A_i^2 + \sigma_n^2] \ .$$

Solving this inequality for σ_n^2 gives

$$\sigma_n^2 \geq \frac{1-\rho_1^2}{\rho_1^2}\sum_{i=1}^{n}\mathbf{E}A_i^2 \ . \tag{31}$$

(If $\rho_1 = 0$, then $\sum_{i=1}^{n}\mathbf{E}A_i^2 = 0$.)

Starting now from $X_i = (X_i + A_i) - A_i$, by the Cauchy-Schwarz inequality, we have

$$\mathbf{E}X_i^2 \leq \mathbf{E}(X_i + A_i)^2 + \mathbf{E}A_i^2 + 2||X_i + A_i||_2||A_i||_2 \ .$$

We sum these inequalities, then we apply Hölder inequality and finally use relations (30) and (31) and some simple calculations to obtain

$$b_n^2 \leq \sum_{i=1}^{n}\mathbf{E}(X_i + A_i)^2 + \sum_{i=1}^{n}\mathbf{E}A_i^2 +$$
$$2\left(\sum_{i=1}^{n}\mathbf{E}(X_i + A_i)^2\sum_{j=1}^{n}\mathbf{E}A_j^2\right)^{1/2} \leq \frac{1+\rho_1}{1-\rho_1}\sigma_n^2 \ .$$

Therefore

$$\sigma_n^2 \geq \frac{1-\rho_1}{1+\rho_1}b_n^2$$

and the lower bound is established.

We shall establish now the upper bound. By simple algebra, Cauchy-Schwarz and Hölder inequalities, we have

$$\begin{aligned}\sigma_n^2 &= -b_n^2 + 2\sum_{i=1}^{n}\mathbf{E}[X_i(X_i + A_i)] \\ &\leq -b_n^2 + 2b_n[\sum_{i=1}^{n}\mathbf{E}(X_i + A_i)^2]^{1/2} \ .\end{aligned}$$

Since for any two positive numbers, a and b, we have $2ab \le (1-\rho_1)^{-1}a^2 + (1-\rho_1)b^2$, we obtain

$$\sigma_n^2 \le \frac{\rho_1}{1-\rho_1} b_n^2 + (1-\rho_1) \sum_{i=1}^{n} \mathbf{E}(X_i + A_i)^2 \ .$$

This last inequality, combined with (30) and solved for σ_n^2 gives

$$\sigma_n^2 \le \frac{1+\rho_1}{1-\rho_1} b_n^2 \ ,$$

and the upper bound is established. $\Diamond$

Remark 14 *Notice that for an independent vector, $\rho_1 = 0$ and Proposition 13 can be viewed as an extension of the classical estimate for variance in the independent case, $\mathbf{E}S_n^2 = \sum_{i=1}^{n} \mathbf{E}X_i^2$.*

As a corollary we obtain the following result in terms of the coefficient of contraction δ defined by (8) that improves the known results in the literature (see for instance Section 1.2.2. in [7] and Proposition 3.2 in [16]).

Corollary 15 *Let $(X_1, X_2, ..., X_n)$ be as in Proposition 13. If $\delta_1 < 1$ then*

$$\frac{1-\delta_1}{(1+\sqrt{\delta_1})^2} \sum_{i=1}^{n} \mathbf{E}X_i^2 \le \mathbf{E}S_n^2 \le \frac{(1+\sqrt{\delta_1})^2}{1-\delta_1} \sum_{i=1}^{n} \mathbf{E}X_i^2 \ .$$

Proof. It was established in Lemma 4.1 in [16] that $\rho_{1,n} < \sqrt{\delta_{1,n}}$.
Then, since the function $(1-x)/(1+x)$ is decreasing, it follows by Proposition 13 that

$$\frac{1-\sqrt{\delta_1}}{1+\sqrt{\delta_1}} b_n^2 \le \mathbf{E}S_n^2 \le \frac{1+\sqrt{\delta_1}}{1-\sqrt{\delta_1}} b_n^2 \ .$$

$\Diamond$

5 Proofs of the main results

5.1 Proof of Theorem 1

We verify the conditions of Proposition 12. Condition (26) follows easily by conditions (11) and (12) combined with Proposition 13 in the following way:

$$\frac{1}{\sigma_n^4} \sum_{j=1}^{n} \mathbf{E}X_{n,j}^4 \le \frac{C_n^2}{\sigma_n^4} \sum_{j=1}^{n} \mathbf{E}X_{n,j}^2 \le \frac{2C_n^2 b_n^2}{\lambda_n b_n^2 \sigma_n^2} = \frac{2C_n^2}{\lambda_n \sigma_n^2} \to 0 \text{ as } n \to \infty. \tag{32}$$

To verify (27) we fix $\varepsilon > 0$ and start from

$$\mathbf{P}(\max_{1 \le j \le n} |A_j| \ge \varepsilon\sigma_n) \le \frac{\mathbf{E}\exp(t \max_{1 \le j \le n} |A_j|)}{\exp t\varepsilon\sigma_n}$$

For $t = \lambda_n/(6C_n)$, by taking into account Lemma 17, we obtain

$$\mathbf{P}(\max_{1\le j\le n} |A_j| \ge \varepsilon\sigma_n) \le (1+\frac{b_n}{3C_n})^2 \exp(-\varepsilon\frac{\lambda_n\sigma_n}{6C_n})$$

and then, (27) follows provided we verify

$$(1+\frac{b_n}{3C_n}) \exp(-\varepsilon\frac{\lambda_n\sigma_n}{12C_n}) \to 0 \text{ as } n\to\infty \ . \tag{33}$$

Notice that assumption (12) implies

$$\frac{\lambda_n\sigma_n}{C_n} \to \infty \text{ as } n\to\infty \ ,$$

that further implies

$$\exp(-\varepsilon\frac{\lambda_n\sigma_n}{12C_n}) \to 0 \text{ as } n\to\infty \ .$$

Moreover

$$\frac{b_n}{C_n}\exp(-\varepsilon\frac{\lambda_n\sigma_n}{12C_n}) = \exp\left(\ln(\frac{b_n}{C_n}) - \varepsilon\frac{\lambda_n\sigma_n}{12C_n}\right)$$

and (33) follows if we show that

$$\ln(\frac{b_n}{C_n}) - \varepsilon\frac{\lambda_n\sigma_n}{12C_n} \to -\infty \ . \tag{34}$$

We write now

$$\ln(\frac{b_n}{C_n}) = \ln(\frac{\lambda_n\sigma_n}{C_n}) + \ln(\frac{b_n}{\lambda_n\sigma_n}) \ .$$

and notice that

$$\ln(\frac{\lambda_n\sigma_n}{C_n}) - \varepsilon\frac{\lambda_n\sigma_n}{24C_n} \to -\infty \ ,$$

so, in order for (34) to hold it is enough to show that for and n sufficiently large

$$\ln(\frac{b_n}{\lambda_n\sigma_n}) \le \varepsilon\frac{\lambda_n\sigma_n}{24C_n} \ .$$

This fact follows if

$$\frac{C_n}{\lambda_n\sigma_n}\ln(\frac{b_n}{\lambda_n\sigma_n}) \to 0 \ . \tag{35}$$

Now, by Proposition 13 we have

$$\frac{\lambda_n^{1/2}}{2} \le \frac{b_n}{\sigma_n} \le \frac{2}{\lambda_n^{1/2}}.$$

So, condition (35) is satisfied provided

$$\frac{\lambda_n\sigma_n}{C_n(1+|\ln(\lambda_n)|)} \to \infty \ .$$

This is exactly the condition that we impose in (12). Thus (27) holds.

We verify (28) by analyzing the variance of both terms involved. For the first term we use Proposition 13 together with conditions (11) and (12) and obtain

$$\frac{1}{\sigma_n^4} var \sum_{j=1}^n X_{n,j}^2 \le \frac{2}{\lambda_n \sigma_n^4} \sum_{j=1}^n \mathbf{E} X_{n,j}^4 \le \tag{36}$$
$$\frac{2C_n^2 b_n^2}{\lambda_n \sigma_n^4} \le \frac{4C_n^2}{\lambda_n^2 \sigma_n^2} \to 0 \text{ as } n \to \infty.$$

To deal with the second term, first we apply Proposition 13 to estimate the variance, then we use condition (11), and finally we take into account the inequality (31) and so,

$$\frac{1}{\sigma_n^4} var \sum_{j=1}^n X_{n,j} A_{n,j} \le \frac{2}{\lambda_n \sigma_n^4} \sum_{j=1}^n \mathbf{E}(X_{n,j}^2 A_{n,j}^2)$$
$$\le \frac{2C_n^2}{\lambda_n \sigma_n^4} \sum_{j=1}^n \mathbf{E} A_{n,j}^2 \le \frac{2C_n^2 \sigma_n^2}{\lambda_n^2 \sigma_n^4} = \frac{2C_n^2}{\lambda_n^2 \sigma_n^2}$$

which converges to 0 under (12). $\Diamond$

5.2 Proof of Corollary 4

This corollary follows from Theorem 1 via a truncation argument.

First construct $\varepsilon_n \to 0$ slowly enough such that condition (15) is still satisfied. We truncate the variables at the level $T_n = \varepsilon_n h(\lambda_n) \sigma_n$, and denote

$$X_{n,i}^{'} = X_{n,i} I(|X_{n,i}| \le T_n) - \mathbf{E} X_{n,i} I(|X_{n,i}| \le T_n)$$

and

$$X_{n,i}^{"} = X_{n,i} - X_{n,i}^{'} \ .$$

We show that the contribution of $\sum_{i=1}^n X_{n,i}^{"}/\sigma_n$ is negligible in $\mathbf{L}_2$ and therefore is negligible for the convergence in distribution. To estimate its variance we apply Proposition 13 and then we take into account the Lindeberg condition (15). We obtain

$$\frac{1}{\sigma_n^2} var(\sum_{i=1}^n X_{n,i}^{"}) \le \frac{2}{\lambda_n \sigma_n^2} \sum_{i=1}^n \mathbf{E}(X_{n,i}^{"})^2$$
$$\le \frac{4}{\lambda_n \sigma_n^2} \sum_{i=1}^n \mathbf{E} X_{n,i}^2 I(|X_{n,i}| > T_n) \to 0 \text{ as } n \to \infty \ .$$

Then, with the notation $(\sigma_n^{'})^2 = var \ \sum_{i=1}^n X_{n,i}^{'}$ we easily derive from the last convergence that

$$\lim_{n \to \infty} \frac{1}{\sigma_n^2} (\sigma_n^{'})^2 = 1 \ .$$

Finally, we apply Theorem 1 to $X^{'}_{n,i}$ with $C_n = 2\varepsilon_n h(\lambda_n)\sigma_n$. We verify (12) by using the definition of $h(\lambda_n)$, since

$$\frac{C_n(1+|\ln(\lambda_n)|)}{\lambda_n\sigma^{'}_n} = 2\varepsilon_n\frac{\sigma_n}{\sigma^{'}_n} \to 0 \text{ as } n\to\infty \ .$$

and the result follows. $\Diamond$

5.3 Proof of Theorem 8

Using our tools we give a short proof of this theorem for completeness. We verify conditions of Proposition 12. Notice that under the assumptions of this theorem Conditions (26) and (28) are verified exactly as in the proof of Theorem 1 by taking into account that $\rho_{n,1} \leq \sqrt{\delta_{n,1}}$ and replacing Proposition 13 by its Corollary 15. The main difference is now that condition (27) follows easily by the estimate

$$||\mathbf{E}(X_{n,k}|\xi_{n,j})||_\infty \leq 2\delta_{n,1}^{k-j}C_n \text{ a.s.}$$

This inequality implies

$$\frac{1}{\sigma_n}||\mathbf{E}(S_n - S_{n,j}|\xi_{n,j})||_\infty \leq \frac{1}{\sigma_n}\sum_{i=j+1}^{n}||\mathbf{E}(X_{n,i}|\xi_{n,j})||_\infty \leq \frac{2C_n}{\alpha_n^{3/2}b_n}$$

which converges to 0 as $n\to\infty$ by condition (17). The theorem is established. $\Diamond$

6 Appendix

In this section we estimate the moments and the exponential moments for the quantity $A_j = A_{n,j} = \mathbf{E}(S_n - S_j|\mathcal{F}_j)$ where S_j are the partial sums associated to a vector of centered random variables $(X_j)_{1\leq j\leq n}$ defined on a probability space $(\Omega,\mathcal{K},P)$, adapted to an increasing filtration of sub-sigma fields of $\mathcal{K}$, $(\mathcal{F}_j)_{1\leq j\leq n}$, $\mathcal{F}_0 = \{\varnothing,\Omega\}$ and ρ_k is defined by (1).

Lemma 16 *Let $p\geq 2$ be a real number and assume the variables have finite moments of order p. Then,*

$$\sum_{j=1}^{n}\mathbf{E}|A_{n,j}|^p \leq 2^{p-2}(\sum_{k=1}^{n-1}\rho_k^{2/p})^p\sum_{i=1}^{n}\mathbf{E}|X_i|^p \ .$$

If for a certain $0<\rho<1$ we have $\rho_k\leq\rho^k$ then, for any $p\geq 2$,

$$\sum_{j=1}^{n}\mathbf{E}|A_{n,j}|^p \leq p^p\frac{1}{4(1-\rho)^p}\sum_{i=1}^{n}\mathbf{E}|X_i|^p \ .$$

Proof. For simplicity, we shall drop the index n from the notation. For j fixed, $1 \le j \le n$, and $1 \le k \le n-j$ let

$$a_k = a_k(j) = \frac{\rho_k^{2/p}}{\sum_{i=1}^{n-j} \rho_i^{2/p}} \ . \tag{37}$$

Notice that $\sum_{k=1}^{n-j} a_k = 1$. By the fact that $x \to |x|^p$ is a convex function, we easily obtain

$$\begin{aligned} |A_j|^p &= |\sum_{i=j+1}^{n} a_{i-j} a_{i-j}^{-1} \mathbf{E}(X_i|\mathcal{F}_j)|^p \le \sum_{i=j+1}^{n} a_{i-j} |a_{i-j}^{-1} \mathbf{E}(X_i|\mathcal{F}_j)|^p \\ &= \sum_{i=j+1}^{n} a_{i-j}^{1-p} |\mathbf{E}(X_i|\mathcal{F}_j)|^p = \sum_{k=1}^{n-j} a_k^{1-p}(j) |\mathbf{E}(X_{j+k}|\mathcal{F}_j)|^p \ . \end{aligned}$$

Then, since $a_k(j) \ge a_k(1)$ and $1-p<0$, it follows

$$|A_j|^p \le \sum_{k=1}^{n-j} a_k^{1-p}(1) |\mathbf{E}(X_{j+k}|\mathcal{F}_j)|^p \ .$$

Next, we use the fact that by the interpolation theory (see Theorem 4.12 in [1]), for $p \ge 2$,

$$\mathbf{E}|\mathbf{E}(X_{j+u}|\mathcal{F}_j)|^p \le 2^{p-2} \rho_u^2 \mathbf{E}|X_{j+u}|^p \ .$$

Combining now these two facts and summing the relations, we obtain,

$$\begin{aligned} \sum_{j=1}^{n-1} \mathbf{E}|A_j|^p &\le 2^{p-2} \sum_{j=1}^{n-1} \sum_{k=1}^{n-j} a_k^{1-p}(1) \rho_k^2 \mathbf{E}|X_{j+k}|^p \\ &\le 2^{p-2} \sum_{k=1}^{n-1} a_k^{1-p}(1) \rho_k^2 \sum_{j=1}^{n} \mathbf{E}|X_j|^p . \end{aligned}$$

By (37) we notice that

$$\sum_{k=1}^{n-1} a_k^{1-p}(1) \rho_k^2 = (\sum_{k=1}^{n-1} \rho_k^{2/p})^p$$

and the first part of this lemma follows.

For proving the second part of this lemma we take into account that a simple computation based on the fact that $1 - x^\beta \ge \beta(1-x)$ for $0 < x \le 1$ and $0 < \beta \le 1$ gives

$$\sum_{k=1}^{n-1} \rho_k^{2/p} = \sum_{k=1}^{n-1} \rho^{2k/p} \le \frac{1}{1-\rho^{2/p}} \le \frac{p}{2} \frac{1}{(1-\rho)} \ ,$$

which combined with the first part of the lemma gives the result. $\Diamond$

For the next lemma we recall the definition (6).

Lemma 17 *Assume that there is $C > 0$ such that $\max_{1\le k\le n}|X_k| \le C$ a.s. and for a certain $0 < \rho < 1$ we have $\rho_k \le \rho^k$. Then, for any nonnegative $t \le (1-\rho)/(6C)$*

$$\mathbf{E}\exp(t\max_{1\le j\le n}|A_j|) \le (1+\frac{2tb_n}{1-\rho})^2 \ .$$

In particular for $t = \frac{1-\rho}{6C}$ we have

$$\mathbf{E}\exp(\frac{1-\rho}{6C}\max_{1\le j\le n}|A_j|) \le (1+\frac{b_n}{3C})^2 \ .$$

Proof. We start the estimate by the Taylor expansion and majorate the maximum term by the sum:

$$\mathbf{E}\exp(t\max_{1\le j\le n}|A_j|) \le 1+\sum_{p=2}^{\infty}\frac{t^p}{p!}\mathbf{E}\max_{1\le j\le n}|A_j|^p + t\mathbf{E}\max_{1\le j\le n}|A_j| \le$$

$$1+\sum_{p=2}^{\infty}\sum_{j=1}^{n}\frac{t^p}{p!}\mathbf{E}|A_j|^p + t\mathbf{E}\max_{1\le j\le n}|A_j| = I + II \ ,$$

where

$$I = 1+\sum_{p=2}^{\infty}\sum_{j=1}^{n}\frac{t^p}{p!}\mathbf{E}|A_j|^p \ .$$

By Lemma 16 and because by the Stirling approximation we have $p^p \le 3^{p-1}p!$ for $p \ge 2$, we obtain

$$\sum_{j=1}^{n}\mathbf{E}|A_j|^p \le \frac{p^p}{4(1-\rho)^p}\sum_{i=1}^{n}\mathbf{E}|X_i|^p \le \frac{3^p p! C^{p-2}b_n^2}{12(1-\rho)^p} \ .$$

Introducing this estimate in the expression of I we have

$$I \le 1+\sum_{p=2}^{\infty}\frac{(3t)^p C^{p-2}b_n^2}{12(1-\rho)^p} = 1+\frac{9t^2b_n^2}{12(1-\rho)^2}\sum_{p=2}^{\infty}\frac{(3tC)^{p-2}}{(1-\rho)^{p-2}} \ .$$

For $t \le \frac{1-\rho}{6C}$ we easily derive

$$I \le 1+\frac{3t^2b_n^2}{4(1-\rho)^2}\left(1-\frac{3tC}{1-\rho}\right)^{-1} \le 1+\frac{3t^2b_n^2}{2(1-\rho)^2}.$$

Moreover, by Lemma 16 it follows that

$$II = t(\mathbf{E}\max_{1\le j\le n}|A_j|) \le t\left(\sum_{j=1}^{n}\mathbf{E}A_j^2\right)^{1/2} \le \frac{tb_n}{1-\rho}$$

and overall

$$I + II \le 1+\frac{3t^2b_n^2}{2(1-\rho)^2}+\frac{tb_n}{1-\rho} \le (1+\frac{2tb_n}{1-\rho})^2$$

and the lemma is established. $\Diamond$

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References

[1] Bradley R. C. (2007). *Introduction to Strong Mixing Conditions.* vol. 1-3, Kendrick Press.

[2] Bradley R. C. (2011). A note on two measures of dependence (manuscript).

[3] Dobrushin, R. (1956). Central limit theorems for non-stationary Markov chains I, II. *Theory of Probab. and its Appl.* **1,** 65-80, 329-383.

[4] Gänssler P. and Häusler E. (1979). Remarks on the Functional Central Limit Theorem for Martingales. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **50**, 237-243.

[5] Hall, P. and Heyde, C. C. (1980). *Martingale limit theory and its application.* Academic Press, New York.

[6] Ibragimov, I. A. (1975). A note on the central limit theorem for dependent random variables. *Theory Prob. Appl.* **20**, 135-141.

[7] Iosifescu, M. and Theodorescu, R. (1969). *Random processes and learning.* Springer, Berlin.

[8] Jones, G. L. (2004). On the Markov chain central limit theorem. *Prob. Surveys* **1,** 299–320.

[9] Kolmogorov, A. N. and Rozanov Yu. A. (1960). On strong mixing conditions for stationary Gaussian processes. *Theor. Probab. Appl.* **5,** 204-208.

[10] Liu, J. S., Wong, W. H., and Kong, A. (1995). Covariance structure and convergence rate of the Gibbs sampler with various scans. *Journal of the Royal Statistical Society*, Series B, **57,** 157-169.

[11] Mengersen, K. and Tweedie, R. L. (1996). Rates of convergence of the Hastings and Metropolis algorithms. *The Annals of Statistics*, **24**, 101-121.

[12] Roberts, G. O. and Rosenthal, J. S. (1997). Geometric ergodicity and hybrid Markov chains. *Electronic Communications in Probability*, **2**, 13-25.

[13] Roberts, G. O. and Tweedie, R. L. (1996). Geometric convergence and central limit theorems for multidimensional Hastings and Metropolis algorithms. *Biometrika*, **83**, 95-11.

[14] Rosenblatt, M. (1971) *Markov Processes, Structure and Asymptotic Behaviour*. Springer-Verlag, Berlin.

[15] Shao, Q. M. (1995). Maximal inequalities for partial sums of ρ-mixing sequences. *Ann. Probab.* **23,** 948-965.

[16] Sethuraman, S. and Varadhan, S. R. S. (2005). A martingale proof of Dobrushin's theorem for non-homogeneous Markov chains. *Electron. J. Probab.* **10,** 1221–1235.

[17] Utev, S. A. (1990). Central limit theorem for dependent random variables, *Prob. Theory and Math. Stat.* Vol. 2, B. Grigelionis et al (eds.), VSP/Mokslas. 519-528.